# Quasi-coherent sheaves (Har II 5, Shaf VI 3.2)

**Def:** A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on a scheme $X$ is quasi-coherent if $X$ has an open affine cover by $U_i = \operatorname{Spec} A_i$ s.t. for each $i$, there is an $A_i$-module $M_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$. $\mathcal{F}$ is coherent if there exists a cover so that each $M_i$ is finitely generated.

**Ex:** Let $i : Y \hookrightarrow X$ be a closed immersion. Then the kernel of $\mathcal{O}_X \twoheadrightarrow i_* \mathcal{O}_Y$ is a sheaf of ideals $\mathcal{I}_Y$, called the ideal sheaf of the closed subscheme $Y$.

For each open affine $U = \operatorname{Spec} A \subseteq X$, $V = i^{-1}(U) = \operatorname{Spec} B$ must be affine, and $\mathcal{I}(U) = \ker(A \to B)$, which is the ideal defining the subscheme $V \hookrightarrow U$. We'll see later that ideal sheaves are quasi-coherent.

## ~ as a functor

If $A$ is a ring, we now see that $M \longmapsto \widetilde{M}$ is a functor from $A$-modules to quasi-coherent sheaves on $\operatorname{Spec} A$. We'll see that this is an equivalence of categories.

First note that for $\mathcal{F}$ quasi-coherent on $X = \operatorname{Spec} A$, we can take a cover by subsets of the form $V = \operatorname{Spec} B$, s.t. $\mathcal{F}|_V \cong \tilde{M}$, which can then be covered by open sets $D(g)$.

The inclusion $D(g) \subseteq V$ corresponds to the ring map $B \to A_g$. Thus, $\mathcal{F}|_{D(g)} \cong \widetilde{M \otimes_B A_g}$, so we can replace the open cover by a cover of distinguished open sets. Since affine schemes are quasi-compact, we can choose only finitely many. So

$$X = D(g_1) \cup \dots \cup D(g_n) \quad \text{s.t.}$$

$\mathcal{F}|_{D(g_i)} = \tilde{M}_i$ for some $A_{g_i}$-module $\tilde{M}_i$.

The following lemma applies familiar properties of localization.

**Lemma:** Let $X = \operatorname{Spec} A$, $\mathcal{F}$ a quasi-coherent sheaf on $X$ and $f \in A$.

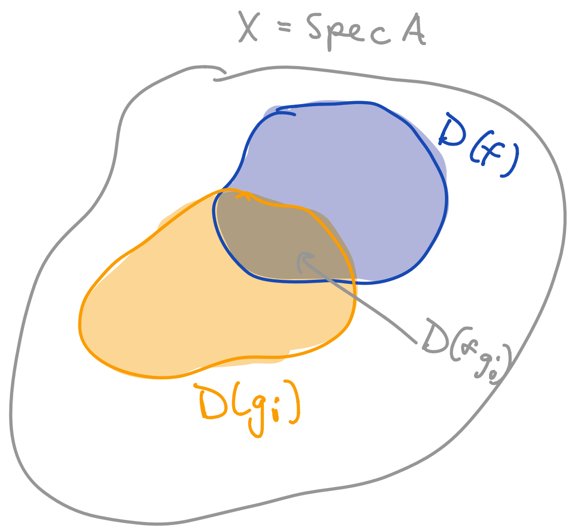
a.) If $s \in \Gamma(X, \mathcal{F})$ s.t. $s \mapsto 0 \in \mathcal{O}(D(f))$, then for some $n > 0$, $f^n s = 0$.

b.) For any $t \in \mathcal{F}(D(f))$, there is some $n > 0$ s.t. $f^n t$ extends to a global section of $\mathcal{F}$.

**Pf:** a.) Let $s \in \Gamma(X, \mathcal{F})$ with $s|_{D(f)} = 0$.

For each $i$, set $s_i = s|_{D(g_i)}$, where $D(g_i)$ form the finite cover described above.

$$D(f) \cap D(g_i) = D(fg_i),$$




so $\tilde{\mathcal{F}}|_{D(fg_i)} = \widetilde{(M_i)_f}$

$\Rightarrow$ the image of $s_i$ is $0$ in $(M_i)_f$.

$\Rightarrow$ $f^n s_i = 0$ for some $n$, so we can choose $n >> 0$ so that it works for every $i$.

Thus, $f^n s$ restricts to $0$ on every open set in an open cover, so $f^n s = 0$.

b.) Let $t \in \tilde{\mathcal{F}}(D(f))$. Again we can restrict it to each $D(fg_i)$, and get $t \in (M_i)_f$.

Thus, $f^n t = t_i \in M_i$ for some $n$. Again, we can choose it to work for every $n$.

On the overlap $D(g_i) \cap D(g_j) = D(g_i g_j)$, we have two sections $t_i, t_j$, which agree on $D(g_i g_j f)$, where they both are $f^n t$.

Thus, applying a.) to $D(g_i g_j)$, we have a section $t_i - t_j$

which is zero on $D(g_i g_j f)$, so $f^m(t_i - t_j) = 0$ on $D(g_i g_j)$ for some $m$. Choose $m \gg 0$ so that it works for all pairs $i, j$.

Then $f^m t_i$ and $f^m t_j$ agree on the overlap, so there is some global section $s$ of $\mathcal{F}$ whose restriction to $D(f)$ is $f^m f^n t = f^{m+n} t$. $\square$

Now we show that if a sheaf on any scheme is quasi-coherent, it'll restrict to some $\tilde{M}$ on <u>every</u> open affine — not just for some open cover. In particular, on an affine scheme, any quasi-coherent sheaf is of the form $\tilde{M}$.

**Prop:** $X$ a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for <u>every</u> open affine $U = \operatorname{Spec} A$, there's an $A$-module $M$ s.t. $\mathcal{F}|_U \cong \tilde{M}$.

**Pf:** ($\Leftarrow$) follows by definition, so let $\mathcal{F}$ be quasi-coherent and $U = \operatorname{Spec} A$ open.

Just as above, we can find a basis for the topology on $X$ consisting of open affines s.t. the restriction of $\mathcal{F}$ to each one is the sheaf associated to a module.

In particular, $U$ is the union of such open sets, so $\tilde{\mathcal{F}}|_U$ is quasi-coherent, so we can reduce to the case $X = \operatorname{Spec} A$ is affine, and let $M = \Gamma(X, \tilde{\mathcal{F}})$.

We define a map $\alpha : \tilde{M} \to \mathcal{F}$ by describing it on distinguished open sets:

If $f \in A$, let $t \in \tilde{M}(D(f)) = M_f$. Then $f^n t \in M = \Gamma(X, \tilde{\mathcal{F}})$, so let $s \in \mathcal{F}(D(f))$ be the image of $f^n t$.

Define $\alpha(t) = \frac{1}{f^n} s$. (Check that this commutes w/ restriction so that it defines a morphism)

$\mathcal{F}$ is quasi-coherent, so we can cover $X$ by open sets $D(g_i)$ w/ $\mathcal{F}|_{D(g_i)} \cong \tilde{M}_i$ for some $A_{g_i}$-module $M_i$. So we get a map

$$\alpha(D(g_i)) : M_{g_i} \longrightarrow M_i .$$

If $a/g_i^m \mapsto 0$, $a \in M$, then the restriction of $a \in \Gamma(X, \tilde{\mathcal{F}})$ to $M_i$ is $0$, so the lemma says $g_i^n a = 0$ for some $n$, so $a/g_i^m = 0$.

If $t \in M_i$, then the lemma says $g_i^n t \in M$ for some $n$, so $\frac{1}{g_i^n}(g_i^n t) \in M_{g_i}$ maps to $t$.

Thus, $\alpha$ induces an isomorphism $M_{g_i} \cong M_i$, so $\alpha|_{D(g_i)}$ is an isomorphism. Since the $D(g_i)$ cover $X$, $\alpha$ is an isomorphism. $\square$

**Remark:** If $X$ is Noetherian, then $\mathcal{F}$ is coherent $\iff$ the same is true as above, only $M$ is a finitely generated $A$-module.

**Cor:** Let $X = \operatorname{Spec} A$. The functor $M \mapsto \tilde{M}$ gives an equivalence of categories between (f.g.) $A$-modules and quasi-coherent (coherent) $\mathcal{O}_X$-modules. It's inverse is $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$.

The global sections functor is left-exact for any sheaf. Given some additional hypotheses on affine schemes, it is exact:

**Prop:** If $X$ is an affine scheme and

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}'$ quasi-coherent, the sequence

$$0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$$

is exact.

Pf: Similar techniques to those used above — see Hartshorne. We'll also be able to prove this easily once we have cohomology by showing $H^1(X, \mathcal{F}') = 0$ in this case.

## More properties of quasi-coherent sheaves:

$X, Y$ schemes, $f: X \to Y$ morphism.

1.) ker, coker, im preserve (quasi-)coherence

2.) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is short exact and

(quasi-) coherent

then $\mathcal{F}$ is (quasi-) coherent.

3.) $\mathcal{G}$ (quasi-)coherent $\Rightarrow$ $f^*\mathcal{G}$ (quasi-)coherent.

4.) If $X$ is Noetherian (or $f$ quasi-compact and separated), then $\mathcal{F}$ quasi-coherent $\Rightarrow$ $f_*\mathcal{F}$ quasi-coherent.

Note: If $\mathcal{F}$ is coherent, $f_*\mathcal{F}$ may not be, even in good circumstances:

Ex: Let $k \hookrightarrow k[x]$ and $f: \mathbb{A}^1_k \to \operatorname{Spec} k$ the corresponding Spec map. Then $\mathcal{O}_{\mathbb{A}^1}$ is coherent on $\mathbb{A}^1$ but $f_*\mathcal{O}_{\mathbb{A}^1} = \widetilde{k[x]}$ on $\operatorname{Spec} k$. But $k[x]$ is not a finitely generated $k$-module.

## Ideal sheaves

Let $i: Y \to X$ be a closed immersion. Recall that the ideal sheaf of $Y$ is

$$\mathcal{I}_Y := \ker(\mathcal{O}_X \to i_*\mathcal{O}_Y)$$

As we alluded to above, we have the following:

**Prop:** Let $Y$ be a closed subscheme of a scheme $X$. Then $\mathcal{I}_Y$ is quasi-coherent on $X$. ($X$ Noetherian $\Rightarrow \mathcal{I}_Y$ coherent.)

**Pf:** The inclusion $i: Y \to X$ is quasi-compact (any preimage of an affine open is again affine and thus quasi-compact). It's also a closed immersion so it's separated. Thus, $i_*\mathcal{O}_Y$ is quasi-coherent, so $\mathcal{I}_Y$ is the kernel of a morphism of quasi-coherent sheaves, so it's quasi-coherent.

If $X$ is Noetherian, then for any open affine $U = \operatorname{Spec} A \subseteq X$, $A$ is Noetherian. Since $\mathcal{I}_Y$ is quasi-coherent, $\mathcal{I}_Y|_U = \tilde{I}$ for $I = \Gamma(U, \mathcal{I}_Y|_U) \subseteq A$, which is finitely generated. So $\mathcal{I}_Y$ is coherent. $\square$

(Moreover, any quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$ is the ideal sheaf of a unique closed subscheme of $X$.)

**Cor**: If $X = \operatorname{Spec} A$, there's a 1-1 correspondence between ideals $I \subseteq A$ and closed subschemes $Y$ of $X$.

**Pf**: If $\mathcal{F} \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals, we know $\mathcal{F} = \tilde{I}$, $I \subseteq A$ an ideal.

If $I \subseteq A$ an ideal, then $\tilde{I} \subseteq \tilde{A} = \mathcal{O}_X$ is a quasi-coherent sheaf of ideals and is thus the ideal sheaf of the closed subscheme $\operatorname{Spec}(A/I) \to X$. $\square$